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AdS/CFT correspondence and D1/D5 systems in theories with 16 supercharges

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Abstract: We discuss spectra of AdS_3 supergravities, arising in the near horizon geometry of D1/D5 systems in orbifolds/orientifolds of type-IIB theory with 16 supercharges. These include models studied in a recent paper (hep-th/0012118), where the group action involves also a shift along a transversal circle, as well as IIB/ΩI_4, which is dual to IIB on $K3$. After appropriate assignements of the orbifold group eigenvalues and degrees to the supergravity single particle spectrum, we compute the supergravity elliptic genus and find agreement, in the expected regime of validity, with the elliptic genus obtained using U-duality map from (4,4) CFTs of U-dual backgrounds. Since this U-duality involves the exchange of KK momentum P and D1 charge N, it allows us to test the (4,4) CFTs in the $P < N/4$ and $N < P/4$ regimes by two different supergravity duals.

Keywords: D-branes, Brane Dynamics in Gauge Theories, AdS-CFT Correspondance.

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Contents

1. Introduction

One of the most interesting examples of AdS/CFT correspondence, proposed in [1], relates type-IIB string theory on $AdS_3 \times S^3 \times M$ to a CFT arising as the infrared fixed point of the effective gauge theory governing a sysytem of Q_1 D1-branes and Q_5 D5-branes. The D5-branes wrap the internal manifold $M = T^4$ or $K3$ and are parallel to the D1-branes, the common world-volume being identified with the boundary of AdS_3. Compared to the higher dimensional cases, testing the correspondence in this case is a more tractable problem, since the CFT's moduli space has points which correspond to exactly solvable theories, given by the symmetric product CFTs M^N/S_N [2]–[6].

Tests of this conjecture were performed in [7]–[9] (for $M = K3$) and [10] (for $M = T^4$), where multiplicities of ground states of the D1/D5 $\mathcal{N} = (4,4)$ CFT were shown to agree with those of (chiral, chiral) primary states in the underlying supergravities. In supergravity the spectrum of (single particle) chiral primaries is determined by group theory via Kaluza-Klein reduction, while multiplicities in the boundary CFT can be read off from the index of the corresponding $\mathcal{N} = (4,4)$ CFT. Although the supergravity description is expected to be valid only for large values of Q_1 and Q_5, the correspondence was shown to work for all $N = Q_1 Q_5$, once a new additive quantum number, the *degree* d, is introduced on the supergravity side [9]. This is a non-negative integer associated to (chiral, chiral) primary states and allows to cut-off multiparticle states and implement the exclusion principle [7]: one keeps only products of chiral primaries whose total degree is $\leq N$.

Adopting the above prescription also for descendants of chiral primaries. i.e. for states of the type (chiral, anything), it was shown in [9] that the multiplicities obtained from supergravity agree, for states of low enough conformal weight $h \leq (N+1)/4$, with the multiplicities obtained from the elliptic genus of the boundary CFT.

It is natural to explore the correspondence between supergravity and boundary CFT in other string theories with sixteen supercharges. The aim is two-fold: on one hand this would provide additional examples of AdS/CFT correspondences in theories with 16 unbroken supercharges. On the other hand, and more importantly, one may hope to be able to learn more about the D1/D5 CFT from the dual supergravity description, especially in cases, as in type-I-like theories, where very little is known about the expected (4,0) CFTs.

In this paper we present a detailed analysis of the spectrum of chiral primary states and their descendants in CFT_2/AdS_3 supergravity pairs, arising from the D1/D5 system in a class of freely acting Z_2 orientifolds of type-IIB theory. Correspondingly, the near horizon geometries are certain freely acting Z_2 orbifolds of $AdS_3 \times S^3 \times T^4$. The associated boundary CFTs have been studied in great detail in [11]. In addition we will consider the AdS_3 supergravity corresponding to type IIB/ΩI_4. Although very little is known about the CFT describing the boundary dynamics in this case, the relevant BPS counting formula can be derived, as we will see, via U-duality from the better understood D1/D5 system in type IIB on $K3$.

The freely acting Z_2 group generators are defined by accompaning the orbifold and/or orientifold actions Ω, I_4, ΩI_4 with a shift σ_{p_6} along a circle transverse to the D1/D5 system, with compact coordinate X^6. We refer to these theories as models I, II and III respectively. The analysis for D1/D5 systems in the presence of a shift σ_{p_1} longitudinal to the world-volume of the D1- and D5-branes, was also performed in the I_4 case (model IV in [11]), but the AdS/CFT dictionary for this system becomes more involve and we postpone a carefull study of it to elsewhere. Besides avoiding technical complications related to the gauge bundles (twisted sectors), these models, with freely acting orbifold group, actions are insteresting in their own right, since D1/D5 states can be mapped via U-duality chains to fundamental string descriptions in terms of type-IIB orbifolds generated by $(-)^{F_L} I_4 \sigma_{p_6}$, $(-)^{F_L} \sigma_{p_6}$ and $I_4 \sigma_{p_6}$ respectively for the first three models ([11, table 1.2]) and to an heterotic background with Wilson lines for the fourth one.

In [11] the effective gauge theories were argued to flow in the infrared to CFTs locally equivalent to the one appearing for the D1/D5 system in type-IIB theory on $T^4 \times S^1$, but with additional Z_2 global identifications induced by the orbifold group actions [11]. The resulting target spaces in the three models are of the form:

$$\mathcal{M}_{\rm higgs} = \left(R^3 \times S^1 \times T^4 \times (T^4)^N / S_N\right) / Z_2 \,. \tag{1.1}$$

The Z_2's are generated by $(-)^{F_L}\, I_4^{\rm c.m.}$, $I_4^{\rm c.m.}\, I_4^{\rm sp}$ and $(-)^{F_L}\, I_4^{\rm sp}$ for the models I, II and III respectively, with $(-)^{F_L}$ the left moving spacetime fermionic number, $I_4^{\rm c.m.}$ the reflection of the first T^4 factor in (1.1) and $I_4^{\rm sp}$ the diagonal Z_2 reflection of the N copies of T^4 in the symmetric product part. Pure D1/D5 systems correspond to states in the untwisted sectors of (1.1). In a similar way, one can read off the BPS multiplicities for the longitudinal shift case (model IV) from (1.1) with the Z_2 induced by I_4, like in model II, but now with states from the twisted sector. The resulting CFTs are of type $\mathcal{N}=(4,4)$ for models II, IV and $\mathcal{N}=(4,0)$ for models I and III, which involve the Ω world sheet parity projection.

The spectrum of BPS charges and multiplicities of D1/D5 excitations can be obtained from the elliptic genus

$$\begin{aligned}\mathcal{Z}\begin{bmatrix}g_0\\h_0\end{bmatrix}(\mathcal{H}_N|q,\bar q,y,\tilde y) &= {\rm Tr}_{\mathcal{H}}\, q^{L_0-c/24}\,\bar q^{\bar L_0-c/24}\,y^{J_0^3}\,\tilde y^{\bar J_0^3}\\ &= \sum C\begin{bmatrix}g_0\\h_0\end{bmatrix}(h,\bar h,\ell,\tilde\ell)\,q^h\,\bar q^{\bar h}\,y^{\ell}\,\tilde y^{\tilde\ell}\end{aligned} \tag{1.2}$$

evaluated in each of the CFT Hilbert spaces $\mathcal{H}_N$ defined by (1.1). The sum in (1.2) runs over $h,\bar h,\ell,\tilde\ell$; $q=e^{2\pi i\tau}$ describe the genus-one worldsheet modulus, $\bar L_0$, L_0 are the Virasoro generators and $\bar J_0^3, J_0^3$ are Cartan generators of an ${\rm SU}(2)_R\times{\rm SU}(2)_L$ current algebra to which the sources y and $\tilde y$ couple respectively. g_0 and h_0 denote the boundary conditions for the various fields appearing in the sigma model. In all the cases results can be written as:

$$\begin{aligned}\mathcal{Z}(p,q,y,\tilde y) &= \sum_N p^N Z(\mathcal{H}_N|q,y,\tilde y)\\ &= \sum_{h=0,\frac12} Z_{cm}\begin{bmatrix}0\\h\end{bmatrix}(q,y,\tilde y)\,\hat Z_F\begin{bmatrix}0\\h\end{bmatrix}(p,y,\tilde y)\hat Z_{sym}\begin{bmatrix}0\\h\end{bmatrix}(p,q,y,\tilde y)\,,\end{aligned} \tag{1.3}$$

where $Z_{cm}\left[{0\atop h}\right](q,y,\tilde y)$ is the contribution to the elliptic genus of the center of mass part in (1.1) and $\hat Z_F\left[{0\atop h}\right](p,y,\tilde y)\hat Z_{sym}\left[{0\atop h}\right](p,q,y,\tilde y)$ the contribution from the symmetric product part. The latter can be computed using [11, (4.17)], which generalize the familiar symmetric product formulas derived in [12]. We have isolated in (1.3) the ground state contribution $\hat Z_F\left[{0\atop h}\right](p,y,\tilde y)$ which admits always perturbative description in terms of U-dual fundamental strings carrying non trivial windings and momenta (see [11, section 2] for details). Indeed, one can easily see that the q^0 coefficient in the expansion of (1.3)

$$\begin{aligned}Z\begin{bmatrix}0\\0\end{bmatrix}(p,y,\tilde y) &= y_-^2\,\tilde y_-^2\,\frac{\vartheta_1^2(y|p)\vartheta_1^2(\tilde y|p)}{\hat\vartheta_1(y\tilde y|p)\hat\vartheta_1(y\tilde y^{-1}|p)\eta^6(p)}.\\ Z_I\begin{bmatrix}0\\1/2\end{bmatrix}(p,y,\tilde y) &= \frac12\,y_-^2\,\tilde y_+^2\,\frac{\vartheta_1^2(\tilde y|p)}{\hat\vartheta_1(y\tilde y|p)\hat\vartheta_1(y\tilde y^{-1}|p)}\,\frac{\vartheta_2^2(y|p)}{\hat\vartheta_2^2(0|p)}\\ Z_{II}\begin{bmatrix}0\\1/2\end{bmatrix}(p,y,\tilde y) &= \frac12\,y_-^2\,\tilde y_-^2\,\frac{\vartheta_2^2(\tilde y|p)}{\hat\vartheta_1(y\tilde y|p)\hat\vartheta_1(y\tilde y^{-1}|p)}\,\frac{\vartheta_2^2(y|p)}{\eta^6(p)}\end{aligned} \tag{1.4}$$

$$Z_{III}\begin{bmatrix}0\\1/2\end{bmatrix}(p,y,\tilde{y}) = \frac{1}{2}\,y_+^2\tilde{y}_-^2\,,\frac{\vartheta_2^2(\tilde{y}|p)}{\hat{\vartheta}_1(y\tilde{y}|p)\hat{\vartheta}_1(y\tilde{y}^{-1}|p)}\frac{\vartheta_1^2(y|p)}{\hat{\vartheta}_2^2(0|p)} \tag{1.5}$$

$$Z_{IV}\begin{bmatrix}1/2\\0\end{bmatrix}(p,y,\tilde{y}) = 16\,y_-^2\tilde{y}_-^2\,\frac{1}{\hat{\vartheta}_1(y\tilde{y})\hat{\vartheta}_1(y\tilde{y}^{-1})\eta^2}\frac{\eta^8}{\vartheta_4^8(0)} \tag{1.6}$$

reproduces, respectively, the multiplicities of fundamental strings in type-IIB theory on $T^4\times S^1/g\sigma_{p_6}$ with g being $(-)^{F_L}I_4$, $(-)^{F_L}$ and I_4 for models I, II, III and the ones for heterotic strings moving on T^5/Z_2^5 for model IV [11].

These results support the CFTs proposals (1.1) at the two-charge level but, as extensively discussed in [11], the predictions coming from (1.3) for the degeneracies of three charge states (D1-D5-KK) are not in agreement with basic constraints imposed by U-duality. In particular, U-duality for $Q_5=1$, maps model II to model III, while exchanging the number of D1-branes $Q_1=N$ with the KK momentum. This in turn means that the elliptic genera of the two systems should be related by $p\leftrightarrow q$ exchange. The problem arises because, even if one trusts just the (4,4) CFT of model II, the $p\leftrightarrow q$ exchange in its elliptic genus does not give an expression admitting a CFT interpretation, assuming that in the CFT a U(1) current algebra should be present. A similar problem afflicts the more familiar $K3$ case: there, string-string duality requires that excitations at level k of the CFT with $N=Q_1$ in $(K3)^N/S_N$ should be mapped to states at level N in a CFT associated to $k=Q_1$ in type IIB/ΩI_4. Again, the elliptic genus (1.3) of $R^4\times(K3)^N/S_N$ does not have, after the $p\leftrightarrow q$ exchange, the definite modular properties required by a CFT interpretation.[1] In [11], we proposed an explanation of this fact, according to which U-duality maps the (4,4) CFTs above to systems with vanishing RR 0- and 4-form backgrounds, whereas the CFT we proposed for model III is valid for non-trivial RR backgrounds, which however cannot be connected continuously to the trivial background. Indeed, in model III only discrete (Z_2) values of RR 0- and 4- form are allowed, since these fields are odd under Ω.

Shedding some light on this issue, from the supergravity point of view, is the basic motivation of the present work. The main result will be that, in fact, the supergravity elliptic genus is, in the expected regime of validity, in agreement with the proposed CFT of model II, but, for both models III and IIB/ΩI_4, it agrees with the predictions of U-duality rather than with the counting formulas coming from the symmetric product CFT proposals in [11]. These results seem to support the arguments given in the conclusions of [11] and recalled in the previous paragraph.

The paper is organized as follows: in section 2 we review the computation of the KK spectrum of $\mathcal{N}=(2,2)$ 6-dimensional supergravity on $AdS_3\times S^3$ and organize it in terms of short multiplets of $\mathrm{SU}(1,1|2)_R\times\mathrm{SU}(1,1|2)_L$. The comparison with boundary CFT is done for (chiral, chiral) states at finite N. In section 3 we extend the

[1] Notice that the CFT in question is the one that should appear in the non supersymmetric sector of a (4,0) CFT.

analysis to our orbifold models which involve (1,1) supergravities in 6-dimensions. We assign Z_2 eigenvalues and degree to the previously obtained short multiplets. Using these assignments, in section 4, we compute the Poincare' polynomial and then the elliptic genus and compare them with those of the CFT s. In section 5, we perform a similar analysis for IIB/ΩI_4, the dual of IIB on $K3$. Finally, in section 6, we give some conclusions.

2. KK-reduction of $D = 6$ $\mathcal{N} = (2,2)$ supergravity on $AdS_3 \times S^3$

In this section we will review the construction of the supergravity multiparticle partition function from the KK data for $\mathcal{N} = (2,2)$ 6-dimensional supergravity, corresponding to type-II theory compactified on T^4. Our analysis will follow essentially [8]–[10].

The KK spectrum can be efficiently organized into short supermultiplets of the AdS_3 supergroup $\mathrm{SU}(1,1|2)_R \times \mathrm{SU}(1,1|2)_L$. This group is generated by the global generators $\{L_{\pm 1}, L_0, G^{\pm 1/2}_{\pm 1/2}, J^3_0, J^{\pm}_0\}_{L,R}$ of the $\mathcal{N} = (4,4)$ Superconformal Algebra (SCA), whose bosonic part is identified with the $\mathrm{SO}(2,2) \times \mathrm{SO}(4)$ isometry group of $AdS_3 \times S^3$. We will denote by $(\mathbf{m = 2j})$ a short supermultiplet, which is a highest weight representation of the chiral SCA, with highest weight state given by the chiral primary state $h = j$. The whole representation is constructed by acting on this state with the raising operators $\{L_{-1}, G^{\pm 1/2}_{-1/2}, J^-_0\}_{L,R}$.

The spectrum of KK harmonics on S^3 can be determined essentially by group theory [13]–[8]. To find the SO(4) (isometry group of the S^3) representations that arise in the KK reduction, one starts from the representations of the little Lorentz group SO(4) under which the 6-dimensional massless states of IIB supergravity, obtained by reducing the 10-dimensional theory on T^4, transform. Identifying the SO(3) tangent group of S^3 as a subgroup of this SO(4), one finds SO(3) representations by decomposing the SO(4) representations under SO(3). Given a field which transforms as spin j under the SO(3) tangent group of S^3, one then expands it using spin j spherical harmonics. The latter in turn transform under the isometry group of S^3 namely $\mathrm{SO}(4) = \mathrm{SU}(2) \times \mathrm{SU}(2)$ as (j_1, j_2) where all j_1 and j_2 appear such that $(j_1 + j_2) \geq j \geq |j_1 - j_2|$. For the first few spin j harmonics one finds results in table 1.[2] For m big enough, we are then left with the following content of KK harmonics coming from six-dimensional massless states in the representations of the $\mathrm{SO}(4) = \mathrm{SU}(2) \times \mathrm{SU}(2)$ little group

$$(2,2) = (0) + (2) + (4) = 3(m,m) + 2(m, m \pm 2) + (m, m \pm 4) \qquad \text{Graviton}$$
$$(2,1) = (1) + (3) = 2(m, m \pm 1) + (m, m \pm 3) \qquad \text{Gravitino}$$
$$(1,1) = (0) + (2) = 2(m,m) + (m, m \pm 2) \qquad \text{Vector}$$

[2] In the following we will always label the SU(2) spin-j representations by specifying integers $2j$.

(2j)	KK-harmonics		
(0)	$(m,m)\quad m=1,2,...$		
(1)	$(m,m\pm 1)\quad m=1,2...$		
(2)	$(m,m)\quad m=2,3,...$	$(m,m\pm 2)\quad m=1,2,..$	
(3)	$(m,m\pm 1)\quad m=2,3...$	$(m,m\pm 3)\quad m=1,2...$	
(4)	$(m,m)\quad m=3,4,...$	$(m,m\pm 2)\quad m=2,3,$	$(m,m\pm 4)\quad m=1,2$

Table 1: S^3 KK-harmonics.

$$
\begin{aligned}
&(2,0)=(2)=(m,m)+(m,m\pm 2) \qquad \text{Self-dual tensor}\\
&(1,0)=(1)=(m,m\pm 1) \qquad \text{Left-spinor}\\
&(0,0)=(0)=(m,m) \qquad \text{Scalar}\,.
\end{aligned} \tag{2.1}
$$

In particular using the field content of $\mathcal{N}=(2,2)$ supergravity in six dimensions (table 2 below) and organizing (m,m') harmonics into short supermultiplets of the $\mathcal{N}=(4,4)$ SCA defined by[3]

$$
(\mathbf{m},\mathbf{m}')=\sum_{i,j=0}^{2}\binom{i}{2}\binom{j}{2}\,(m-i,m'-j)\,, \tag{2.2}
$$

one is left with the following spectrum of one-particle supergravity states [8]

$$
\begin{aligned}
\mathcal{H}_{\text{single particle}} = \oplus_{m\geq 1}\,[(\mathbf{m},\mathbf{m+2})+(\mathbf{m+2},\mathbf{m})+4(\mathbf{m},\mathbf{m+1})+\\
+4(\mathbf{m+1},\mathbf{m})+6(\mathbf{m+1},\mathbf{m+1})]+5(\mathbf{1},\mathbf{1})\,.
\end{aligned} \tag{2.3}
$$

At this point, on the supergravity side there is no notion of finite N physics. de Boer associated a new quantum number, the *degree*, to the (chiral, chiral) primaries in order to truncate the supergravity states in a systematic way and reproduce the (chiral, chiral) states of the finite N symmetric product boundary CFT. We recall the argument here since it will be useful for us later in order to identify the Z_2 actions on the supergravity side. One starts from the Poincare' polynomial on the CFT side which captures the information about the chiral primaries. Under spectral flow, chiral primaries in the NS sector go to the ground states of the Ramond sector. The generating function for the Ramond ground states has already been discussed in the last section. Flowing back from the Ramond states to the NS states, using the relations (here we use the convention that a Ramond ground state carries zero dimension)

$$
h_R=h_{NS}-j_{NS} \qquad j_R=j_{NS}-\frac{c}{12} \tag{2.4}
$$

[3]In the following the letters m,m' will denote individual components of the $\mathcal{N}=4$ supermultiplets while block letters $\mathbf{m},\mathbf{m}'$ will denote the entire supermultiplets labelled by their chiral primaries.

the generating function of the Poincare' polynomials for symmetric products of T^4 is given by

$$P = \prod_{m=0}^{\infty} \prod_{r,s=0}^{2} (1 - p^{m+1} y^{m+r} \tilde{y}^{m+s})^{-(-1)^{r+s} h_{r,s}} \,, \tag{2.5}$$

where $h_{r,s}$ are the Hodge numbers of the torus. Explicitly $(-1)^{r+s} h_{r,s} = d(r) \cdot d(s)$ where $d(0) = d(2) = 1$ and $d(1) = -2$. The coefficient of p^N in the expansion of P, in the limit $N \to \infty$ is given by the residue of P at $p = 1$. Indeed, P has a first order pole, coming from the $m = r = s = 0$ term, and the residue, P_∞, is then given by:

$$P_\infty = \frac{(1-y)^2 (1-\tilde{y})^2}{(1-y^2)(1-\tilde{y}^2)(1-y\tilde{y})^5} \prod_{m=1}^{\infty} \frac{(1-y^{m+1}\tilde{y}^m)^4 (1-y^m \tilde{y}^{m+1})^4}{(1-y^{m+1}\tilde{y}^{m+1})^6 (1-y^{m+2}\tilde{y}^m)(1-y^m\tilde{y}^{m+2})} \,. \tag{2.6}$$

This is exactly the content of the (chiral, chiral) states in the bulk supergravity as seen from (2.3), except for the states which involve either j or j' equal to zero. The missing states are precisely $(0,1), (1,0), (0,2)$ and $(2,0)$ states. It has been argued [8, 9] that these states do not correspond to propagating degrees of freedom in the bulk, but neverthless appear at the boundary of AdS_3. Including these states on the supergravity side, one finds a complete matching of the chiral primaries in the supergravity and the large N limit of CFT. Note however that the $(0,0)$ state (corresponding to the identity operator) which is responsible for the simple pole in the CFT, does not appear in the supergravity side.

In order to extend this equality to finite N, de Boer introduced a notion of degree. To each (chiral, chiral) primary (m, m'), one asscociates a degree $d(m, m')$ which couples to the variable p in such a way that the supergravity Poincare' polynomial reproduces exactly the CFT one. The $(4,4)$ SCA is assumed to commute with the degree and, as a result, all the descendants of a (c,c) primary carry the same degree as the primary itself. Thus the supergravity single particle Hilbert space is described as

$$\mathcal{H}_{\text{single particle}} = \oplus'_{m \geq 0} \, h_{r,s} \, (\mathbf{m+r}, \mathbf{m+s})_{m+1} \,, \tag{2.7}$$

where the subscript $m+1$ denotes the degree and the $\oplus'$ means $(\mathbf{0}, \mathbf{0})$ is omitted from the sum. In the above equation we have also added $(\mathbf{0}, \mathbf{1})$, $(\mathbf{1}, \mathbf{0})$, $(\mathbf{0}, \mathbf{2})$ and $(\mathbf{2}, \mathbf{0})$ supermultiplets to the supergravity single particle states. While $(\mathbf{0}, \mathbf{2})$ and $(\mathbf{2}, \mathbf{0})$ supermultiplets contain the higher modes of the left and right $\mathcal{N} = 4$ super Virasoro generators, $(\mathbf{0}, \mathbf{1})$ and $(\mathbf{1}, \mathbf{0})$ contain the left and right $\mathrm{U}(1)^4$ super Kac-Moody generators. The multiparticle Hilbert space is obtained as usual by taking products of single particle states with the degrees being an additive quantum number. The finite N CFT Hilbert space is then conjectured to be

$$\mathcal{H}_N^{CFT} = \mathcal{H}^{\text{multiparticle}}|_{\text{degree} \leq N} \,. \tag{2.8}$$

By construction, the right-hand side reproduces the multiplicities of the (c,c) primaries of the CFT. Note that the $(1-p)^{-1}$ factor of the CFT appears in the gravity side due to the fact that one allows all states with degree up to (and not just equal to) N.

3. Supergravity spectra for type IIB orbifolds/orientifolds in the presence of transverse shifts

In this section we will implement the free Z_2 orbifold/orientifold (models I, II, III introduced above), on the KK spectrum of 6-dimensional supergravities on $AdS_3\times S^3$ obtained in the previous section. If we consider the radius R_6 of the circle, along which the shift is performed, very large, in such a way that the space transverse to the D1/D5 system is effectively R^4, then the near horizon geometry will still be $AdS_3 \times S^3 \times T^4$. However, in doing the KK reduction the various modes will come with non-trivial Z_2 phases due to the orbifold group actions (Ω, I_4 and ΩI_4 according to the model) in the way we will specify below.

The relevant Z_2-eigenvalues for 6-dimensional massless fields of $\mathcal{N} = (2, 2)$ supergravity, together with their transformation properties under the little group SO(4), are displayed in table 2. For example, the first row in table 2 is the contribution of the 6-dimensional metric $G_{\mu\nu}$, the RR two-form $B^R_{\mu\nu}$, the dilaton and the scalars associated to the internal components G_{ij}, B^{RR}_{ij} of the metric and RR two-form. They are clearly even under all three orbifold group actions. One can similarly obtain the other contributions with the corresponding eigenvalues. As expected, one sees from the above table that the model II has (2,0) supersymmetry while models I and III have (1,1) supersymmetry in 6-dimensions.

One now proceeds as in the T^4 case, decomposes these fields into representations of the SO(3) tangent group of S^3 (which is the diagonal subgroup of the SU(2)×SU(2) little group). At this point the information of the chirality is lost and one finds that all the three models have the same representation content of SO(3) and, as a result, will have the same S^3 spherical harmonics. One would then naively conclude that all the three models are identical on the supergravity side. The subtle point however is that for a given S^3 spherical harmonic, the dimensions L_0 are different for the three models. To compute the dimensions one would have to solve the equations of

I	II	III		
Ω	I_4	ΩI_4	bosons	fermions
$+$	$+$	$+$	(2,2)+(0,2)+(2,0)+17(0,0)	2(1,2) + 10(1,0)
$-$	$+$	$-$	4(0,2)+4(2,0)+8(0,0)	2(1,2)+ 10(1,0)
$+$	$-$	$-$	8(1,1)	2(2,1)+ 10(0,1)
$-$	$-$	$+$	8(1,1)	2(2,1) + 10(0,1)

Table 2: SO(4) field content with Z_2 eigenvalues.

motion for various fields. However, given the fact that all the systems at least have (4,0) supersymmetry, which fixes the $\bar{L}_0$ eigenvalues in terms of the data encoded in the spherical harmonics, we can deduce the L_0 eigenvalues by determining the spin $s = L_0 - \bar{L}_0$, as has been done in [8]. But before doing so, we give an intuitive argument to determine the Z_2 actions on the different representations.

Let us consider the SO(3) spin 2 and 3/2 states. Spin 2 appears only from the graviton (i.e. from the first row), while spin 3/2 appears from gravitini, and therefore 2 from each row in table 2. The spherical harmonics of spin 2 and spin 3/2, among others, would contain $(m, m+4)$ and $(m, m+3)$ states respectively. Now $(m, m+4)$ appears in the short multiplet $(\mathbf{m}, \mathbf{m+2})$. This short multiplet also contains 4 states of the type $(m, m+3)$, two of which appear by applying left supersymmetry generator $G^i_{-1/2}$ once and the other two by applying two left supersymetry generators and one right supersymmetry generator $G^1_{-1/2}G^2_{-1/2}\tilde{G}^i_{-1/2}$. The remaining 4 $(m, m+3)$ states appear in the 4 short multiplets $(\mathbf{m}, \mathbf{m+1})$ by applying 2 left supersymmetry generators. The 4 $(m, m+3)$ states that appear in the short multiplet $(\mathbf{m}, \mathbf{m+2})$ must come from the first two rows in the above table. This is because in model II the Z_2 action must commute with the (4,4) supersymmetry and it is the first two rows that come with positive Z_2 eigenvalue. The remaining 4 $(m, m+3)$ states appearing in the 4 short multiplets $(\mathbf{m}, \mathbf{m+1})$ must therefore come from the last two rows. In particular, it also implies that in model II, the short multiplets $(\mathbf{m}, \mathbf{m+2})$ must appear with + eigenvalues and $(\mathbf{m}, \mathbf{m+1})$ must appear with $-$ eigenvalue. Having identified the 4 $(m, m+3)$ states appearing in the short multiplet $(\mathbf{m}, \mathbf{m+2})$ as the ones coming from the first two rows, it now becomes clear that in models I and III the Z_2 actions would not commute with (4,4) supersymmetry. In fact, they only commute with (4,0) supersymmetry, with the left supersymmetry generator $G^i_{-1/2}$ having negative eigenvalues. This argument also shows that of the 4 $(\mathbf{m}, \mathbf{m+1})$ short multiplets that appear from the last two rows, the (c,c) primaries of the two should come with + eigenvalue and two with $-$, the role of + and $-$ being exchanged between models I and III. Furthermore we conclude that while $\tilde{G}^i_{-1/2}$ moves the states within each row, $G^i_{-1/2}$ moves vertically between first two rows as well as the last two rows.

We can repeat the above procedure for all the states and using the fact that the Z_2 of model II commutes with (4,4) supersymmetry, show that all the short multiplets of the type $(\mathbf{m+r}, \mathbf{m+s})$ with $r+s$ even come from the first two rows and all the ones with $r+s$ odd come from the last two rows. Furthermore, the (c,c) primaries of these short multiplets appear in the following way: $(m, m+2)$, $(m+2, m)$ and two (m, m) appear in the first row, 4 (m, m) appear in the second row, and in the third and fourth row each 2 $(m, m+1)$ and 2 $(m+1, m)$.

We will now justify the above result by explicitly computing the L_0 eigenvalues. We will decompose the 6-dimensional (2,0) or (1,1) supermultiplets into (1,0) supermultiplets. The latters, upon KK reduction on $AdS_3 \times S^3$, give rise to (4,0) supermultiplets of $\mathrm{SU}(1,1|2)_R \times \mathrm{SU}(2)_L \times \mathrm{SU}(1,1)_L$, which can be labelled by $(\mathbf{m}, m'; h)$ where $\mathbf{m}$ denotes the $2J_3$ eigenvalue of the chiral primary of the $\mathrm{SU}(1,1|2)_R$ superalgebra, $m'/2$ and h denote the isospin and L_0 eigenvalue under the left $\mathrm{SU}(2)_L \times \mathrm{SU}(1,1)_L$ subgroup of the isometry of $S^3 \times AdS_3$. Even though there is no supersymmetry in the left sector, h is obtained by determining the helicity as in [8]. The method is as follows. Given a field which transforms as (n_1, n_2) under the little group $\mathrm{SU}(2) \times \mathrm{SU}(2)$ (which is not the same as the isometry of S^3), its S^3 harmonic labelled by (m, m') under the isometry group carries, a helicity s which is given by all possible values of $y_1 - y_2$ subject to the condition $y_1 + y_2 = m' - m$, with y_1 and y_2 being the J_3 component of the SU(2) in the representations (n_1) and (n_2) respectively.

This analysis has been done in [8] for graviton, vector, tensor and hypermultiplets. We label these multiplets as $\mathbf{G}$, $\mathbf{T}$, $\mathbf{V}$ and $\mathbf{H}$ respectively. The result is

$$\begin{aligned}
\mathbf{G}: \quad & \left(\mathbf{m}, m+2; \frac{m+2}{2}\right) + \left(\mathbf{m}, m; \frac{m}{2}\right) + \left(\mathbf{m}, m; \frac{m+4}{2}\right) + \\
& + \left(\mathbf{m}, m-2; \frac{m+2}{2}\right) + \left(\mathbf{m}, m-2; \frac{m-2}{2}\right) + \left(\mathbf{m}, m-2; \frac{m}{2}\right) \\
\mathbf{T}: \quad & \left(\mathbf{m}, m; \frac{m}{2}\right) + \left(\mathbf{m}, m-2; \frac{m+2}{2}\right) \\
\mathbf{V}: \quad & \left(\mathbf{m}, m; \frac{m+2}{2}\right) + \left(\mathbf{m}, m-2; \frac{m}{2}\right) \\
\mathbf{H}: \quad & 2\left(\mathbf{m}, m-1; \frac{m+1}{2}\right).
\end{aligned} \tag{3.1}$$

Note that the tensor and vector multiplets, although they come with same spherical harmonics, carry different L_0 eigenvalues. Besides these standard (1,0) multiplets, we also need two multiplets with highest spin 3/2 which appear in the decomposition of (2,0) and (1,1) representations in terms of (1,0) representation. These multiplets (say $\mathbf{U}$ and $\tilde{\mathbf{U}}$) contain the following representations under the little group:

$$\mathbf{U} = (1,2) + 2(0,2)\,, \qquad \tilde{\mathbf{U}} = (2,1) + 2(1,1) + (0,1) \tag{3.2}$$

KK analysis for these two multiplets give:

$$\begin{aligned}
\mathbf{U}: \quad & \left(\mathbf{m}, m-3; \frac{m-1}{2}\right) + \left(\mathbf{m}, m-1; \frac{m+1}{2}\right) + \left(\mathbf{m}, m+1; \frac{m+3}{2}\right) \\
\tilde{\mathbf{U}}: \quad & \left(\mathbf{m}, m-3; \frac{m+1}{2}\right) + \left(\mathbf{m}, m-1; \frac{m+3}{2}\right) + \left(\mathbf{m}, m-1; \frac{m-1}{2}\right) + \\
& + \left(\mathbf{m}, m+1; \frac{m+1}{2}\right).
\end{aligned} \tag{3.3}$$

To illustrate how we obtained the above multiplets let us consider $\mathbf{U}$. The (1,2) and (0,2) give rise to following components

$$(1,2): \qquad \left(m-2, m+1; h-\frac{1}{2}, h\right) + \left(m, m+1; h-\frac{3}{2}, h\right) +$$

$$
\begin{aligned}
&+\left(m-2,m-1;h+\frac{1}{2},h\right)+\left(m,m-1;h+\frac{1}{2}\pm 1,h\right)+\\
&+\left(m-2,m-3;h+\frac{3}{2},h\right)+\left(m,m-3;h+\frac{1}{2},h\right)\\
2(0,2):\qquad &2\left(m-1,m+1;h-1,h\right)+2\left(m-1,m-1;h,h\right)+\\
&+2\left(m-1,m-3;h+1,h\right). \qquad (3.4)
\end{aligned}
$$

Here the first and second entries are twice the spin under the right and left SU(2)s and the third and fourth entries are the $\bar{L}_0$ and L_0 eigenvalues. h at this stage is undetermined but $\bar{h}$ is given in terms of h plus the helicity. The fact that these states must organize in at least (4,0) supermultiplets, fixes the values of h and one gets the result (3.3). It is interesting to note that even in the non-supersymmetric sector h satisfies the bound $h \geq m'/2$!!

Using the fact that the (2,0) gravitational and tensor multiplets are given by

$$
\mathbf{G}_{(\mathbf{2},\mathbf{0})} = \mathbf{G} + 2\mathbf{U}\,, \qquad \mathbf{T}_{(\mathbf{2},\mathbf{0})} = \mathbf{T} + \mathbf{H} \qquad (3.5)
$$

we find that the dimensions h are such that the (4,0) representations combine to form (4,4) representations as expected. The result is

$$
\begin{aligned}
\mathbf{G}_{(\mathbf{2},\mathbf{0})} &= (\mathbf{m},\mathbf{m}+\mathbf{2}) + (\mathbf{m},\mathbf{m}) + (\mathbf{m},\mathbf{m}-\mathbf{2})\\
\mathbf{T}_{(\mathbf{2},\mathbf{0})} &= (\mathbf{m},\mathbf{m})\,. \qquad (3.6)
\end{aligned}
$$

On the other hand the (1,1) gravitational and vector multiplets are

$$
\mathbf{G}_{(\mathbf{1},\mathbf{1})} = \mathbf{G} + 2\tilde{\mathbf{U}} + \mathbf{T}\,, \qquad \mathbf{V}_{(\mathbf{1},\mathbf{1})} = \mathbf{V} + \mathbf{H}\,. \qquad (3.7)
$$

It is easy to see from eqs.(3.1), (3.3) that the dimensions h are such that the right hand side of (3.7) cannot be organized as (4,4) multiplets. Indeed in $\mathbf{V}_{(\mathbf{1},\mathbf{1})}$ there is no state with $h = m'/2$. In fact for all the states in $\mathbf{V}_{(\mathbf{1},\mathbf{1})}$, $h = (m'+1)/2$. Therefore formally we can express the (1,1) vector multiplet as the first left descendents of (4,4) multiplets:

$$
2\mathbf{V}_{(\mathbf{1},\mathbf{1})} = (\mathbf{m},\mathbf{m}+\mathbf{1})^- + 2(\mathbf{m},\mathbf{m})^- + (\mathbf{m},\mathbf{m}-\mathbf{1})^-\,, \qquad (3.8)
$$

where the superscript $-$ indicates that we should keep only the states with odd number of $G_{-1/2}$ acting on (c,c) primary. Similarly

$$
\begin{aligned}
\mathbf{G}_{(\mathbf{1},\mathbf{1})} = {} & (\mathbf{m},\mathbf{m}+\mathbf{2})^+ + 2(\mathbf{m},\mathbf{m}+\mathbf{1})^+ + 2(\mathbf{m},\mathbf{m})^+\\
& +2(\mathbf{m},\mathbf{m}-\mathbf{1})^+ + (\mathbf{m},\mathbf{m}-\mathbf{2})^+\,, \qquad (3.9)
\end{aligned}
$$

where the superscript $+$ indicates that we should keep only states with even number of $G_{-1/2}$ acting on the (c,c) primary. Note that the superscript $\pm$ are in fact the Z_2

assignments to the various (4,4) multiplets appearing in the parent IIB theory on T^4, and the correlation of $\pm$ with even and odd $G_{-1/2}$ indicates that Z_2 anticommutes with $G_{-1/2}$. This is exactly the result we obtained from the earlier intuitive argument. The fact that in (1,1) supergravity the states cannot be organized in terms of complete (4,4) multiplets is in agreement with string theory description of IIA on $AdS_3 \times S^3 \times K3$ [5], where it is shown that only (4,0) supersymmetry survives.

Accepting the above composition of the short multiplets, we still have to assign degree to them. This has already been done for the T^4 as in eq. (2.7), but here we need more refinement. We need to assign degrees for (c,c) primaries coming from each row separately. For example, there are altogether 6 (c,c) primaries of the form (m, m), 4 coming from the second row while 2 from the first row of table 2. Given the fact that they should be organized as 1 (m, m), 4 $(m+1, m+1)$ and 1 $(m+2, m+2)$ having all degree $m+1$, with $m \geq 0$, we note there should be only 5 $(1,1)$ primaries. In the first row, indeed, there is one less $(1,1)$ compared to general (m, m)s. This allows us to deduce that the $(m+2, m+2)$ series comes from the first row. Since (m, m) series corresponds to $h_{0,0}$ one would expect that for all the three models this should be in the spectrum, therefore it is natural to assume that it too comes from the first row. Finally, we are left with 4 $(m+1, m+1)$, which should all be in the second row. A more delicate question is regarding 4 $(m, m+1)$ and $4(m+1, m)$ type (c,c) primaries. From the T^4 analysis we know that there should be 2 $(m, m+1)$ and 2 $(m+1, m+2)$ each with degree $m+1$ which should be distributed between third and fourth rows. We do not know of any a priori reason to choose a particular assignment, but again anticipating the agreement with the U-dual fundamental theory, we assign 2 $(m, m+1)$ in the third row and 2 $(m+1, m+2)$ in the fourth row. Similarly, we assign 2 $(m+2, m+1)$ in the third row and the 2 $(m+1, m)$ in the fouth row, each with degree $m+1$.

To conclude, we have the following Z_2 assignments for various models.

$$\mathcal{H}^A_{\text{single particle}} = \oplus'_{m\geq 0}\, h_{r,s}\, (\mathbf{m+r}, \mathbf{m+s})^{\epsilon_A(r,s)}_{m+1}\,, \tag{3.10}$$

where A=I,II,III, the subscript $m+1$ denotes the degree and $\epsilon_A(r,s)$ are the Z_2 eigenvalues of the multiplets and are given by:

$$\epsilon^I(r,s) = (-1)^s\,, \qquad \epsilon^{II}(r,s) = (-1)^{r+s}\,, \epsilon^{III}(r,s) = (-1)^r\,. \tag{3.11}$$

For models I and III, although we have grouped the states in terms of the original (4,4) multiplets, we have to remember that $G^i_{-1/2}$ anticommutes with the Z_2's, and therefore the descendants that involve odd numbers of $G^i_{-1/2}$'s will appear with an extra minus sign under the Z_2 action.

Let us now try to understand the Z_2 actions defined in eqs.(3.10), (3.11) as geometric actions on T^4. The degree 1 i.e. $m = 0$ corresponds to a single copy of T^4. The (c,c) primaries are generated by the two chiral left moving and right

JHEP03(2001)035

moving fermions ψ_i and $\tilde{\psi}_i$ $(i = 1, 2)$. The $h_{r,s}$ numbers of (r, s) (c,c) primaries are then simply $\tilde{\psi}^r \psi^s$ where we have suppressed the subscripts i, j on the fermions. The Z_2 actions in (3.10), (3.11) imply that g_I reflects ψ, g_{II} reflects both ψ and $\tilde{\psi}$ and g_{III} reflects $\tilde{\psi}$. One can now deduce the Z_2 action on the bosonic fields ∂X and $\bar{\partial} X$, which are the superpartners of ψ and $\tilde{\psi}$ respectively, from its action on supersymmetry generators G and $\tilde{G}$. The result can be summarized as $g_I = (-1)^{F_L}$, $g_{II} = I_4$ and $g_{III} = (-1)^{F_L}.I_4$. These are indeed the Z_2 actions proposed in [11] for the three symmetric product CFT s.

Before proceeding further, we would like to comment on a puzzle. Note that for model I and III, the graviton and vector multiplets have to be assigned the degrees in different ways. In particular for model I

$$\mathbf{V}_{(\mathbf{1},\mathbf{1})} = \sum_{m=0}^{\infty} \sum_{r=0}^{2} |d(r)| (\mathbf{m}+\mathbf{r}, \mathbf{m}+\mathbf{s})^{-}_{m+1}; \qquad s = 1$$
$$\mathbf{G}_{(\mathbf{1},\mathbf{1})} = = \sum_{m=0}^{\infty} \sum_{s=0,2} \sum_{r=0}^{2} |d(r)| (\mathbf{m}+\mathbf{r}, \mathbf{m}+\mathbf{s})^{+}_{m+1} \tag{3.12}$$

with similar expressions for model III but with the role of "r" and "s" exchanged. Although the above choice is required to obtain the correct Poincare' polynomials as dictated by U-duality, it is a bit puzzling why these two models, which are both 6-dimensional (1,1) supergravity theories, give rise to different results. However, one can see that the difference lies in the following two facts:

1. the definition of degree, whose understanding itself is outside the realm of supergravity. Indeed in each of these models, for $m \geq 1$, two of the (c,c) primaries $(m + 1, m)$ come with positive eigenvalue and the remaining two with negative eigenvalue under the Z_2 action. For model I, we assign degree $m + 2$ to the two positive eigenstates and $m + 1$ to the negative eigenstates, and vice versa for model III. Similarly, for (c,c) primaries $(m, m+1)$ for $m \geq 1$ there are again two positive and two negative eigenstates. For model I, the degree of the positive eigenstates is $m + 1$ and that of the negative eigenstates is $m+2$, and vice versa for model III. We do not have any a priori reason for this assignment, however the relation between the three Z_2 s namely $g_I.g_{II} = g_{III}$ fixes the degree for model III once one assumes it for model I.

2. the two $(\mathbf{0}, \mathbf{1})$ and two $(\mathbf{1}, \mathbf{0})$ are assigned eigenvalues minus and plus respectively for model I and vice versa for model III. These multiplets contain the left and right $\mathrm{U}(1)^4$ super Kac-moody generators respectively. They in fact do not represent propagating degrees of freedom in the bulk and therefore they are not contained in the supergravity spectrum. They should appear as large gauge transformations that do not vanish at the boundary of AdS_3, in the same way as the $(\mathbf{2}, \mathbf{0})$ and $(\mathbf{0}, \mathbf{2})$ multiplets arise from large super-diffeomorphisms.

In the 6-dimensional theory obtained by compactifying IIB on T^4, there are 16 U(1) gauge fields with 4 each coming from the reduction of the metric, NS-NS anti-symmetric tensor, and RR 2- and 4-forms. Of these only 8 gauge transformations should be non-vanishing at the AdS_3 boundary, giving rise to $\mathrm{U}(1)_L^4 \times \mathrm{U}(1)_R^4$ current algebra. The Z_2 actions we have defined here indicate that these 8 gauge fields are the ones coming from the T^4 reduction of the metric and the RR 2-form. Indeed we have assigned 2 $(1,0)$ and 2 $(0,1)$ chiral primaries in the third and the fourth rows of table 2 respectively. The U(1) currents are the first descendents of the chiral primary. Using now the fact that $\tilde{G}_{-1/2}$ and $G_{-1/2}$ move the states horizontally and vertically, respectively, between the third and the fourth rows, it is easy to see that all the U(1) currents appear in the third row. The 6-dimensional vectors which appear in the third row carry positive eigenvalues under Ω and therefore must come from the metric and the RR 2-form.

One may wonder if this could be understood directly from a supergravity analysis. However 4 T-dualities on T^4 exchanges model I and III, so it would appear that if in model I, the metric and RR 2-forms give rise to the U(1) currents, then in model III the NS-NS 2-form and RR 4-form should generate these currents. This would be in contradiction with our proposal here. The point is, that under the T-duality, Q_1 and Q_5 get exchanged. Therefore, more precisely our proposal is, that for $Q_1 \gg Q_5$, the U(1) currents come from the metric and RR 2-form. Supergravity analysis on the other hand depends only on the product $Q_1 \cdot Q_5$ and not the ratio Q_1/Q_5 therefore it seems unlikely that the answer can be found there. The world sheet approach [5] for fundamental string moving in $AdS_3 \times S^3 \times T^4$, corresponding to the near horizon geometry of Q_5 NS5 branes and Q_1 fundamental strings, depends on the individual values of Q_1 and Q_5. In this framework the weakly coupled string (for a fixed order one volume of T^4) corresponds to precisely the regime $Q_1 \gg Q_5$. The U(1) currents on the AdS_3 boundary come from the NS-NS sector of the string theory and therefore should appear from the bulk gauge fields arising from metric and NS-NS 2-forms.[4] This is exactly the S-dual version of our proposal here.

4. Comparison between supergravity and CFT elliptic genera

Now we can construct the multiparticle Hilbert space $\mathcal{H}_{\text{multiparticle}}$ as usual and identify the finite N CFT Hilbert space with the subset of states in $\mathcal{H}_{\text{multiparticle}}$ that have degree less than or equal to N. The generating function of the Poincare' polynomials,

[4] In fact an attempt to find the U(1) currents corresponding to the RR gauge fields has not succeeded so far [15].

in the sector which is projected by Z_2, is

$$\frac{1}{1-p}\mathrm{Tr}_{(c,c)\in\mathcal{H}_{\text{multiparticle}}} g^A p^{m+1} y^\ell \tilde{y}^{\bar{\ell}}\,, \tag{4.1}$$

where $m+1$, ℓ and $\bar{\ell}$ are, respectively, the degree, and the left- and right-moving values of twice the SU(2) spins. g^A are the generators of the various Z_2's. The prefactor $1/1-p$ just takes into account the fact that finite N CFT Hilbert space is identified with states in $\mathcal{H}_{\text{multiparticle}}$ that have degree up to N. One can carry out the trace over all (c,c) primaries above with the result

$$\prod_{m=0}^{\infty}\prod_{r,s=0}^{2}(1-\epsilon^A(r,s)p^{m+1}y^{m+r}\tilde{y}^{m+s})^{-(-)^{r+s}h_{r,s}}\,. \tag{4.2}$$

It is easy to see that these exactly reproduce the CFT results (1.5) for the three models under spectral flow from Ramond to NS sector. Note that although models I and III we have only (4,0) supersymmetry, they arise as untwisted sector of orbifold of (4,4) theory and we can use the spectral flow in the parent (4,4) theory. More generally we can still define a notion of spectral flow using the SU(2) level n current algebra in the non-supersymmetric sector. By spectral flow from "Ramond" to "NS" in the non-supersymmetric sector we then mean the map from spin-j to spin-$((n/2)-j)$ characters. The unitarity of the "Ramond" sector CFT then implies the bound $h_{NS}\geq j_{NS}/2$ in the "NS" sector. The states contributing to the Poincare' polynomials are the ones which saturate this bound and therefore correspond to the ground states in the "Ramond" sector.

One can now, following de Boer, check the correspondence beyond the (c,c) primaries. The idea is to construct the finite N elliptic genus which is obtained by taking the trace over states of the form chiral on the right-moving sector and any state on the left-moving sector, and setting $\tilde{y}=\bar{q}^{-1/2}$. The comparison of the states can, of course, only be made for dimensions much less than N, since otherwise gravity approximation would break down. In [8] de Boer showed that for the K3 case the matching of the states goes all the way upto left dimension equal to $(N+1)/4$. This is exactly the bound at which black hole is expected to form. In the right-moving sector arbitrary chiral states are allowed, and they have a bound on the dimension which is of order $N/2$. Setting $\tilde{y}=\bar{q}^{-1/2}$ for right-chiral states, one actually looses the information about which chiral state is being traced over. One might ask a more refined question of matching the states between CFT and supergravity for a fixed right chiral state. One can show that this does not work even in the $K3$ case. The point is that there could be states of the form (m,m'), 2 $(m,m'-1)$ and $(m,m'-2)$, where m denotes an arbitrary left-moving state, while the states appearing on the right-moving sector m', $m'-1$ and $m'-2$ are all chiral states. Setting $\tilde{y}=\bar{q}^{-1/2}$ the contribution of these 4 states to the supertrace vanishes. Therefore the fact that the states on the two sides do not match for arbitrary $\tilde{y}$ implies that discrepancy comes in the combinations of the above 4 states. On the other hand, these 4 states can in principle combine to form a long multiplet on the right-moving sector. So, it appears that as one changes the parameters of the theory from a region where CFT is valid to the region where gravity approximation is valid, some of such combinations of chiral states become non-chiral and form long multiplets. It would be interesting to understand the precise mechanism of how this happens. However, in the following we will set $\tilde{y} = \bar{q}^{-1/2}$ and compare the elliptic genus on the two sides.

In model I, unfortunately the elliptic genus is zero, but here one can repeat the analysis of [10] and take 2 derivatives with respect to $\tilde{y}$ before setting it to $\bar{q}^{-1/2}$. In this case however we do not get any information; in fact for states satisfying the bound $h \leq (N+1)/4$ only the ground state contributes as shown in [10].

Models II and III are more interesting, since for $\tilde{y} = \bar{q}^{-1/2}$ the elliptic genus does not vanish. The elliptic genus for CFT has already been obtained in section 4 of [11], where the appropriate Z_2 actions for models II and III have been used. The elliptic genus for supergravity is computed as follows. We start from the supertrace over the states in the single particle Hilbert space which are of the form chiral in the right-moving sector and arbitrary state from the left-moving one. There will be two sectors, one where the identity element of Z_2 is inserted and the second where the generator g^A is inserted.

$$Z(p,q,y) \equiv \frac{1}{2}\mathrm{tr} p^n q^m y^\ell \equiv \sum_{n,m,\ell} \frac{1}{2}\Big[c^{A+}_{\rm sgr}(n,m,\ell) + c^{A-}_{\rm sgr}(n,m,\ell)\Big] p^n q^m y^\ell$$
$$Z^A(p,q,y) \equiv \frac{1}{2}\mathrm{tr} g^A p^n q^m y^\ell \equiv \sum_{n,m,\ell} \frac{1}{2}\Big[c^{A+}_{\rm sgr}(n,m,\ell) - c^{A-}_{\rm sgr}(n,m,\ell)\Big] p^n q^m y^\ell\,, \quad (4.3)$$

where n, m and ℓ are the degree, left dimension and twice the left J_3 quantum numbers respectively. The elliptic genus for the multi-particle states is then

$$Z^A_{\rm multiparticle}(p,q,y) = \frac{1}{1-p}\prod_{n,m,\ell}(1-p^n q^m y^\ell)^{-c^{A+}_{\rm sgr}(n,m,\ell)} \cdots (1+p^n q^m y^\ell)^{-c^{A-}_{\rm sgr}(n,m,\ell)}\,, \quad (4.4)$$

where again $(1-p)^{-1}$ is included for the same reason as in (4.1). This factor can actually be absorbed inside the product by redefining $c^{A+}_{\rm sgr}(n,m,\ell) \to c^{A+}_{\rm sgr}(n,m,\ell) + (1/2)\delta_{n1}\delta_{m0}\delta_{\ell 0}$ which is equivalent to $Z \to Z + (p/2)$ and $Z^A \to Z^A + (p/2)$. In the following Z, Z^A and c's will always denote these redefined quantities.

$Z(p,q,y)$ vanishes for $\tilde{y} = \bar{q}^{-1/2}$ since the elliptic genus for T^4 case is 1. The fact that the trace with identity insertion vanishes implies $c^{A+}_{\rm sgr}(n,m,\ell) = -c^{A-}_{\rm sgr}(n,m,\ell)$ The elliptic genus for the multi-particle states is then

$$Z^A_{\rm multiparticle}(p,q,y) = \prod_{n,m,\ell}\left[\frac{1+p^n q^m y^\ell}{1-p^n q^m y^\ell}\right]^{c^{A+}_{\rm sgr}(n,m,\ell)}\,. \quad (4.5)$$

To compute the c's we write explicitly the trace in (4.3) as

$$Z^A(p,q,y) = \frac{1}{2}\sum_{m,r,s}\sum_{t=0}^{min(2,m+s)}\sum_{k=0}^{\infty}{}' d(r)d(s)d^A(t)\epsilon^A(r,s)p^{m+1}q^{\frac{m+s+t}{2}+k}\times$$
$$\times\sum_{j=0}^{m+s-t} y^{m+s-t-2j}\,, \tag{4.6}$$

where we have used the fact that $(-1)^{r+s}h_{r,s} = d(r)\cdots d(s)$ with $d(0) = d(2) = 1$ and $d(1) = -2$. The sum over m, r, s takes into account all (c,c) primaries $(m+r, m+s)$. We have also included here the $m = r = s = 0$, which is not a (c,c) primary in supergravity, to take into account the redefined Z's including a shift by $p/2$. The sum over k takes into account the descendents coming from applying L_{-1} and the primed sum over k means that for $m+s = 0$ there is only one term in the sum, namely $k = 0$, and for $m+s \neq 0$ the sum is over all the non-negative integers. This is due to the fact that for $m+s = 0$ we have the left ground state and L_{-1} annihilates this state. The sum over j takes care of the descendents coming from applying J_- and finally sum over t takes into account the descendents coming from applying $G^i_{-1/2}$. The upper bound on this sum means that if $m+s$ is less than 2, then we can only apply a maximum of $m+s$ $G^i_{-1/2}$'s to the chiral primary. $d^A(t)$ takes into account the multiplicities of these descendents, together with the Z_2 actions. Since in model II, the Z_2 commutes with (4,4) supersymmetry, $d^{II}(t) = d(t)$. On the other hand for models I and III, the Z_2's anticommute with $G^i_{-1/2}$'s and therefore $d^I(t) = d^{III}(t) = d(t)$ for t even and $d^I(1) = d^{III}(1) = -d(1) = 2$. Note that since $\epsilon^I(r,s) = (-1)^s$ the sum over r yields zero on the right hand side as expected for model I. For models II and III, since $\epsilon(r,s)$ contains $(-1)^r$, the summation over r gives a factor of 4. Equation (4.6) reduces to:

$$Z^{II}(p,q,y) = 2pq\frac{(1+p)^2}{(1-q)(y-y^{-1})}\left[y^3\frac{(1-q^{1/2}y^{-1})^2}{1-pq^{1/2}y} - y^{-3}\frac{(1-q^{1/2}y)^2}{1-pq^{1/2}y^{-1}}\right]+$$
$$+2p+2\frac{p^2+2p}{1-q}(q^{1/2}(y+y^{-1})-2q) \tag{4.7}$$
$$Z^{III}(p,q,y) = 2pq\frac{(1-p)^2}{(1-q)(y-y^{-1})}\left[y^3\frac{(1+q^{1/2}y^{-1})^2}{1-pq^{1/2}y} - y^{-3}\frac{(1+q^{1/2}y)^2}{1-pq^{1/2}y^{-1}}\right]+$$
$$+2p+2\frac{p^2-2p}{1-q}(q^{1/2}(y+y^{-1})+2q)\,. \tag{4.8}$$

Expanding the above expressions in power series in p^nq^m it is clear that $c^{A+}_{\rm sgr}(n,m,\ell) = 0$ for $m < n/4$ except for $(n,m) = (1,0)$ and in this case $c^{A+}_{\rm sgr}(1,0,0) = 2$ which gives rise to a double pole in (4.5) at $p = 1$ together with a factor $(1+p)^2$ in the numerator. In fact for $m > 0$ the only value of m for which $c^{A+}_{\rm sgr}(4m,m,\ell) \neq 0$ is for $m = 1/2$.

This means that the elliptic genus is of the form

$$Z^A_{\text{multiparticle}}(p,q,y) = \sum_{m,\ell} \frac{1}{(1-p)^2} P_{m\ell}(p) q^m y^\ell \,, \tag{4.9}$$

where $P_{m\ell}$ is a polynomial in p of degree $4m+2$ (after including the $(1+p)^2$ factor in the numerator of (4.5)). We can express

$$\frac{P_{m\ell}(p)}{(1-p)^2} = \frac{a_{m\ell}}{(1-p)^2} + \frac{b_{m\ell}}{1-p} + P'_{m\ell}(p) \,, \tag{4.10}$$

where $P'_{m\ell}$ is a polynomial of degree $4m$. This means that if we restrict ourselves to $m < n/4$, then only the coefficients $a_{m\ell}$ and $b_{m\ell}$ contribute to the elliptic genus.

The same applies also to the CFT side. We start from the Ramond elliptic genus for $\tilde{y} = 1$ which is given by

$$Z^A_{\text{cft}}(p,q,y) = \prod_{n,m,\ell} \left[\frac{1 + p^n q^m y^\ell}{1 - p^n q^m y^\ell} \right]^{c^{A+}_{\text{cft}}(mn,\ell)} \,, \tag{4.11}$$

where we have used the fact that for $\tilde{y} = 1$ the partition function for the identity sector vanishes. $c^{A+}_{\text{cft}}(m,\ell)$ are given by $\frac{1}{2}\text{tr} g^A q^m y^\ell$ where the supertrace is taken over the (4,4) CFT with target space T^4 in zero momentum and winding sectors. g^{II} is I_4 and g^{III} is $(-1)^{F_L} I_4$. It follows that

$$\begin{aligned} \sum_{m,\ell} c^{II+}_{\text{cft}}(m,\ell) q^m y^\ell &= 8 \left[\frac{\theta_2(q,z)}{\theta_2(q,0)} \right]^2 \\ \sum_{m,\ell} c^{III+}_{\text{cft}}(m,\ell) q^m y^\ell &= -8 \left[\frac{\theta_1(q,z)}{\theta_2(q,0)} \right]^2 \,, \end{aligned} \tag{4.12}$$

where $y = e^{2\pi i z}$. Using the fact that in both these models, the U(1), which couples to y, comes with a current algebra which commutes with the Z_2 action and contributes to the total stress energy tensor via a Sugawara term, we conclude that

$$c^{A+}_{\text{cft}}(m,\ell) \equiv c^{A+}_{\text{cft}}(4m - \ell^2) \tag{4.13}$$

is only a function of $(4m - \ell^2)$. Furthermore $c^{A+}_{\text{cft}}(4m - \ell^2)$ vanish when the argument is less than -1 and satisfy, for model II

$$\begin{gathered} \sum_\ell c^{II+}_{\text{cft}}(4m - \ell^2) = 8\delta_{m0} \,, \qquad \sum_\ell \ell c^{II+}_{\text{cft}}(4m - \ell^2) = 0 \,, \\ c^{II+}_{\text{cft}}(0) = 2 c^{II+}_{\text{cft}}(-1) = 4 \,, \end{gathered} \tag{4.14}$$

and for model III

$$\begin{gathered} \sum_\ell c^{III+}_{\text{cft}}(4m - \ell^2) = 0 \,, \qquad \sum_\ell \ell c^{III+}_{\text{cft}}(4m - \ell^2) = 0 \,, \\ c^{III+}_{\text{cft}}(0) = -2 c^{III+}_{\text{cft}}(-1) = -4 \,. \end{gathered} \tag{4.15}$$

In the NS sector the elliptic genus is obtained by the spectral flow and is given by the same expression as (4.5), with the exponent replaced by $c^{A+}_{\rm cft}(4mn-n^2-\ell^2)$. For both models, for $m>0$ the latter are nonvanishing only for $n\le 4m$ and for $m=0$, n must be 1 and $\ell=0$. For $m=\ell=0$ and $n=1$ using the above equation we find that in both the models the elliptic genus have double poles at $p=1$ and there is a factor of $(1+p)^2$ in the numerator. We can now use the same arguments as in the supergravity case to conclude that for dimensions less than $N/4$ only contribution comes from the coefficients of the double pole and the single pole at $p=1$.

Thus the matching of CFT states and supergravity states for dimensions less than $N/4$ implies that the coefficients of the double pole and the single pole at $p=1$ should match on the two sides. This in turn implies

$$\begin{aligned}\sum_n c^{A+}_{\rm cft}(4mn-n^2-\ell^2) &= \sum_n c^{A+}_{\rm sgr}(m,n,\ell)\\ \sum_n n c^{A+}_{\rm cft}(4mn-n^2-\ell^2) &= \sum_n n c^{A+}_{\rm sgr}(m,n,\ell)\,.\end{aligned} \tag{4.16}$$

We first compute these quantities in the supergravity side. The two quantities that we are interested in are just Z^A and the first derivative of Z^A with respect to p evaluated at $p=1$. From (4.7), (4.8) after some algebra one can show that for model II

$$\begin{aligned}Z^{II}(1,q,y) &= -14-2\frac{q^{1/2}(y+y^{-1})+2q}{1-q}+\frac{8}{1-q^{1/2}y}+\frac{8}{1-q^{1/2}y^{-1}}\\ \frac{\partial Z^{II}(p,q,y)}{\partial p}|_{p=1} &= 2+\frac{8q^{1/2}y}{(1-q^{1/2}y)^2}+\frac{8q^{1/2}y^{-1}}{1-q^{1/2}y^{-1}}\end{aligned} \tag{4.17}$$

and for model III

$$\begin{aligned}Z^{III}(1,q,y) &= 2-2\frac{q^{1/2}(y+y^{-1})+2q}{1-q}\\ \frac{\partial Z^{III}(p,q,y)}{\partial p}|_{p=1} &= 2\,.\end{aligned} \tag{4.18}$$

In order to evaluate these quantities on the CFT side we define t and u via $m=t/2$ and $\ell=t-2u$ with t and u non-negative integers so that $c^{A+}_{\rm cft}(4mn-n^2-\ell^2)=c^{A+}_{\rm cft}(4u(t-u)-(n-t)^2)$. We can now sum over n using equations (4.14), (4.15), taking care of the ranges of n. For $|\ell|\ge 2$, the sum over n from 1 to infinity includes all the terms in (4.14) and (4.15), while for $|\ell|=1$, $n=0$ term is missing and for $\ell=0$, $n=0$ term as well as $n=-1$ (for $m=0$) are missing. The result for model II is

$$\begin{array}{lll} & \sum_n c^{II+}_{\rm cft}(4mn-n^2-\ell^2) & \sum_n n c^{II+}_{\rm cft}(4mn-n^2-\ell^2)\\ |\ell|\ge 2 & 8\delta_{m,|\ell|/2} & 8|\ell|\delta_{m,|\ell|/2}\\ |\ell|=1 & 8\delta_{m,1/2}-2 & 8\delta_{m,1/2}\\ \ell=0 & 6\delta_{m,0}-4 & 2\delta_{m,0}\end{array} \tag{4.19}$$

and for model III

$$\begin{array}{lcc} & \sum_n c_{\rm cft}^{III+}(4mn-n^2-\ell^2) & \sum_n n c_{\rm cft}^{III+}(4mn-n^2-\ell^2) \\ |\ell|\geq 2 & 0 & 0 \\ |\ell| = 1 & -2 & 0 \\ \ell = 0 & 4-2\delta_{m,0} & 2\delta_{m,0} \end{array} \tag{4.20}$$

Expanding the supergravity results (4.17), (4.18), one finds complete agreement between supergravity and CFT for model II. For model III however, we find that all the numbers agree except for $\ell = 0$ and $m > 0$ for which supergravity gives -4 while CFT gives $+4$. To see this descrepancy, let us look at this case more closely. For $\ell = 0$

$$\sum_{n=1}^{\infty} c_{\rm cft}^{III+}(4mn-n^2) = \sum_{n\in\mathcal{Z}} c_{\rm cft}^{III+}(4m^2-(n-2m)^2) - c_{\rm cft}^{III+}(0) - \delta_{m,0} c_{\rm cft}^{III+}(-1)\,. \tag{4.21}$$

For $m = 0$ the $c_{\rm cft}^{III+}(0) = -4$ corresponds to the multiplicities of the ground states and this certainly agrees with supergravity since this information already entered in the Poincare' polynomial. For $m > 0$ CFT still gives $c_{\rm cft}^{III+}(0) = -4$ since they depend only on the combination $(4m - \ell^2)$, while the supergravity result indicates that it should be $+4$, which happens to be the value of $c_{\rm cft}^{II+}(0)$. A CFT with U(1) current algebra which couples to y certainly cannot have this property. We have already observed this mismatch of $c_{\rm cft}^{III+}(0)$ for $m = 0$ and $m > 0$, when discussing the U-duality predictions for the 3-charge system. U- duality says that the elliptic genus for model III should be the same as model II with p and q exchanged. Let us see if the supergravity result agrees with the model II after exchanging p and q. From equation (4.11), which describes the symmetric product part of the CFT, we see that this is

$$\prod_{n=1}^{\infty}\prod_{m=0}^{\infty}\prod_{\ell}\left[\frac{1+p^nq^my^\ell}{1-p^nq^my^\ell}\right]^{c^+(mn,\ell)}, \tag{4.22}$$

where $c^+(m,\ell)$ are given by

$$\begin{aligned} c^+(0,\ell) &= c_{\rm cft}^{III+}(-\ell^2)\,, && \text{for}\, m = 0 \\ c^+(m,\ell) &= c_{\rm cft}^{II+}(4m-\ell^2)\,, && \text{for}\, m > 0\,. \end{aligned} \tag{4.23}$$

In (4.22) we have also included the contribution of the center of mass CFT ($R^4 \times T^4/I_4$) for model II, which, after exchanging p and q is only a function of p. This in fact gives the $m = 0$ terms in (4.22). By exchanging p and q in the symmetric product part we would also have obtained terms in the product with $n = 0$, but these terms just reproduce the elliptic genus of the center of mass CFT for model III, which the supergravity side cannot see. Therefore, we have dropped these terms. $c^+(m,\ell)$ defined above has exactly the property that it is -4 for $m = \ell = 0$ and

+4 for $m = \ell^2/4 > 0$. Indeed, by repeating the previous analysis for the elliptic genus (4.22), we find complete agreement with the supergravity result for model III. We will make some comments on this remarkable fact in the conclusions.

5. Type-I' supergravity on $AdS_3 \times S^3$

In this section we determine the supergravity spectrum for type IIB/ΩI_4, further compactified on $AdS_3 \times S^3$. We do not have a CFT description of the D1/D5 system in this theory. In this paper we will not attempt to find this directly, like in the models discussed in the previous section, but rather use supergravity to extract informations about the D1/D5 system in this background. As already mentioned in the introduction, one expects this D1/D5 system to be related to the better understood D1/D5 system in type IIB on $K3$. To be more precise, the map between orbifold group generators and charges in the two backgrounds is given in the table 3.

For a single D5-brane, $Q_5 = 1$, since the D1 charge is associated to the power of p while the KK charge is associated to the power of q in the expansion of the elliptic genus, table 3 implies that a counting formula for excitations in the D1/D5 system in type IIB/ΩI_4 can be obtained from the type IIB on $K3$ result after $p \leftrightarrow q$ exchange. We are then led to the following prediction for the D1-D5-KK multiplicities in IIB/ΩI_4:

$$\tilde{Z}(p,q,y,\tilde{y}) = y_-^2 \tilde{y}_-^2 \tilde{Z}_{cm}(q,y,\tilde{y})\, \tilde{Z}_F(p,y,\tilde{y}) \tilde{Z}_{sp}(p,q,y,\tilde{y})\,, \tag{5.1}$$

where

$$\begin{aligned} \tilde{Z}_{cm}(q,y,\tilde{y}) &= \frac{1}{\hat{\vartheta}_1(y\tilde{y}|q)\hat{\vartheta}_1(y\tilde{y}^{-1}|q)\eta^{18}(q)} \\ \tilde{Z}_{sp}(p,q,y,\tilde{y}) &= \prod_{n,m=1}^{\infty} (1 - p^n q^m y^l \tilde{y}^{\tilde{l}})^{-c_{K3}(nm,l,\tilde{l})} \end{aligned} \tag{5.2}$$

both come from the symmetric product piece in the dual $R^4 \times (K3)^N/S_N$ and

$$\tilde{Z}_F(p,y,\tilde{y}) = \frac{\hat{\vartheta}_1^2(\tilde{y}|p)}{\hat{\vartheta}_1(y\tilde{y}|p)\hat{\vartheta}_1(y\tilde{y}^{-1}|p)} \tag{5.3}$$

C	$\xrightarrow{S}$	**D**	$\xrightarrow{T_{15}}$	**E**	$\xrightarrow{S}$	**F**
D_1		F_1		p_1		p_1
D_{12345}		NS_{12345}		NS_{12345}		D_{12345}
p_1		p_1		F_1		D_1
I_4		I_4		$(-)^{F_L} I_4$		ΩI_4

Table 3: IIB on $K3$ versus type IIB/ΩI_4.

together with the zero mode factor $y_-^2\tilde{y}_-^2$ is the contribution of the center of mass R^4. Notice that unlike in the more familiar cases of type IIB on T^4 or $K3$, $Z_F(p,y,\tilde{y})$ is not the naive extension to $p=0$ of $\tilde{Z}_{sp}(p,q,y,\tilde{y})$ and therefore (5.1) does not admit a description in terms of a CFT partition function. We can however compare (5.1) with the prediction from supergravity on $AdS_3\times S^3$, in following the logic of the previous section for the freely acting orbifold model III.

The study of chiral primary states in the AdS reduction of the present supergravity theory follows closely our analysis for model III in the previous section, with some important differences. First of all, the $(4,0)$ supermultiplets with negative eigenvalue under ΩI_4 are now projected out already at the level of one-particle states. This should be contrasted with the case where the ΩI_4 orbifold group action is accompained with a shift and supermultiplets with both eigenvalues enter in the computation although counted with different signs. Indeed, in the absence of a shift, the boundary theory is no longer a Z_2 orbifold of the parent one in type IIB on T^4, and looks (even locally) drastically different from it. The second important difference is the presence in this case of open string sectors living on D5-branes. They lead to 16 additional vector multiplets of the $\mathcal{N}=(1,1)$ six-dimensional supersymmetry. The (4,0) multiplet content together with degree for the vector multiplets is given in (3.12).

To keep our expressions as compact as possible and our discussion close to our previous analysis, we will insist in organizing states in terms of "$(4,4)$ supermultiplets", with $(4,0)$ supermultiplets inside them taken with alternate ΩI_4 eigenvalues. The restriction to states with even ΩI_4 eigenvalues will be performed only at the end by keeping only descendants with even(odd) numbers of $G^i_{-1/2}$ acting on a given even(odd) chiral primary ground state. In fact in (3.12) the vector multiplets are already expressed in terms of odd (4,4) multiplets. Using eq. (3.12) for n_V-4 vector multiplets (in the present case $n_V=20$) living on D5-branes together with the results for untwisted states from model III, we are then left with the one-particle supergravity Hilbert space

$$\mathcal{H}^A_{\text{single particle}} = \oplus'_{m\geq 0}\,\tilde{h}_{r,s}\,(\mathbf{m}+\mathbf{r},\mathbf{m}+\mathbf{s})^{\epsilon_{III}(r,s)}_{m+1} \tag{5.4}$$

with

$$(-1)^{r+s}\tilde{h}_{r,s} = d(r)\left[d(s)-\frac{n_V-4}{2}\delta_{s,1}\right] \tag{5.5}$$

and $\epsilon_{III}=(-)^s$. As before, we read off the spectrum of (chiral, chiral) primary states from the supergravity Poincare' polynomial, which after keeping even chiral primaries in (5.4) reduces to:

$$P_\infty = \frac{(1-y)^2}{(1-y^2)(1-\tilde{y}^2)(1-y\tilde{y})}\prod_{m=1}^{\infty}\frac{(1-y^{m+1}\tilde{y}^m)^2(1-y^m\tilde{y}^{m+1})^2}{(1-y^{m+1}\tilde{y}^{m+1})^2(1-y^{m+2}\tilde{y}^m)(1-y^m\tilde{y}^{m+2})}\,, \tag{5.6}$$

where we have added the usual missing multiplets $2(1,0)+(2,0)+(0,2)$. One can easily see that this agrees with the residue of the first order pole at $p=1$ of (5.3), after a spectral flow to the NS sector.

The spectrum of descendants of the form (chiral, anything) is now determined from

$$\tilde{Z}(p,q,y) = \sum_{m,r,s} \sum_{t=0}^{min(2,m+r)} {\sum_{k=0}^{\infty}}' \tilde{h}^{r,s}(-)^s \left(\frac{1+(-)^{t+s}}{2}\right) p^{m+1} q^{(m+r+t/2)+k} \times \\ \times \sum_{j=0}^{m+r-t} y^{m+r-t-2j} ,$$

where the projector $((1+(-)^{t+s})/2)$ has been inserted in the trace in order to ensure that only states with ΩI_4 eigenvalue $+1$ enter in the sum. After a straightforward algebra, one finds

$$\tilde{Z}(p,q,y) = 2pq\frac{(1-p)^2}{(1-q)(y-y^{-1})}\left[\frac{y^3+yq+(n_V/2)y^2q^{1/2}}{1-pq^{1/2}y} - \right. \\ \left. -\frac{y^{-3}+qy^{-1}+(n_V/2)q^{1/2}y^{-2}}{1-pq^{1/2}y^{-1}}\right] + 2p + 2\frac{p^2-2p}{1-q}\left(q^{1/2}(y+y^{-1})+\frac{n_V}{2}q\right). \tag{5.7}$$

In particular

$$\tilde{Z}(1,q,y) = 2 - \frac{2}{1-q}\left[q^{1/2}(y+y^{-1})+\frac{n_V}{2}q\right] \\ \frac{\partial\tilde{Z}(p,q,y)}{\partial p}|_{p=1} = 2 \tag{5.8}$$

leading to

	$\sum_n c^{I'}_{\text{sugra}}(m,n,l)$	$\sum_n nc^{I'}_{\text{sugra}}(m,n,l)$
$\lvert\ell\rvert \geq 2$	0	0
$\lvert\ell\rvert = 1$	-2	0
$\ell = 0$	$(n_V+2)\delta_{m,0} - n_V$	$2\delta_{m,0}$

(5.9)

One can easily see that these results for $n_V = 20$ are in complete agreement with $\sum_n c_{\text{CFT}}(m,n,l)$ and $\sum_n nc_{\text{CFT}}(m,n,l)$, with $c_{CFT}(n,m,l)$ the coefficients coming from the $K3$ symmetric product expression after $p \leftrightarrow q$ exchange i.e. $c_{CFT}(m,l) = c_{K3}(m.l)$ for $m \neq 0$; $c_{CFT}(0,0) = -2c_{CFT}(0,\pm 1) = -4$.

Therefore supergravity agrees with U-duality even in this case.

6. Conclusions

In this paper we studied the 6-dimensional (2,0) and (1,1) supergravity theories on $AdS_3 \times S^3$ arising from orbifolding/orientifolding of IIB theory on T^4. These theories arise in certain type-II or type-I compactifications. While (2,0) theories give rise to

(4,4) superconformal symmetry $\mathrm{SU}(1,1|2)_R \times \mathrm{SU}(1,1|2)_L$, the (1,1) theories only have (4,0) symmetry $\mathrm{SU}(1,1|2)_R \times \mathrm{SU}(1,1)_L \times \mathrm{SU}(2)_L$. In the case of freely acting orbifolds the Poincare' polynomials encoding the multiplicities of (chiral,chiral) primaries in the AdS supergravity theories were computed and shown to agree with the predictions coming from D1/D5 CFTs proposed in [11]. We also computed the supergravity Poincare' polynomial for type I', which agrees with the pure D1/D5 bound state multiplicities as predicted by U-duality. We extended the supergravity analysis to the elliptic genus, which should encode the information about the multiplicities of D1/D5/KK bound states. In the (4,4) case this agrees with the elliptic genus for the (4,4) symmetric product CFT in the expected range of validity, namely $P < Q_1/4$, where P is the KK momentum. On the other hand, in the (4,0) case the supergravity elliptic genus does not agree with the one for the symmetric product (4,0) CFT proposed in [11]. Instead it agrees, in the range of validity, with the multiplicities obtained from the elliptic genus of the U-dual (4,4) CFT via $P \leftrightarrow Q_1$ exchange. Therefore, quite remarkably, the (4,4) CFT is probed by gravity duals in both $P < Q_1/4$ as well as $Q_1 < P/4$ ranges; in the former range the gravity dual being the usual (4,4) one, while in the latter it is the (4,0) AdS_3 supergravity. The results therefore provide a very strong evidence for the correctness of the proposed (4,4) symmetric product CFTs.

Although we have focussed here on the models considered in [11] and the standard type-I' model, one can easily generalize the methods given here to other models. To give an example, consider 6-dimensional (2,1) supergravity obtained by comactifying IIB on T^4/Z_2, where the Z_2 is acting asymmetrically only on the left movers together with a shift in a transverse direction. We can do the KK analysis on $AdS_3 \times S^3$ exactly as in section 3. The resulting spectra organize themselves in (4,4) multiplets as in eq. (3.10), with the Z_2 action being given by $(-1)^r$. We can go on to compute the elliptic genus, and in the regime of validity, the two relevant moments of expansion coefficients are given by eq. (4.18), with the only difference being that the sign of the $2q$ term in the numerator of the right hand side of the first equation is reversed. The (4,4) symmetric product CFT which agrees with this supergravity is the one corresponding to $(T^4)^N/S_N$ modded by the diagonal Z_2 action which acts on each copy of T^4 asymmetrically only on the right-moving sector.

However several important questions remain to be answered. For example, what is the correct (4,0) boundary CFT which is dual to the (4,0) AdS_3 supergravity? The analysis reported here shows that this CFT can not be of a symmetric product type. In [11] it has been shown that there is no (4,0) CFT which completely reproduces the elliptic genus of the U-dual (4,4) symmetric product CFT after $P \leftrightarrow Q_1$ exchange. What we are asking here, however, is a weaker condition, namely, we are looking for a (4,0) boundary CFT which reproduces the elliptic genus of the (4,0) AdS_3 supergravity in the region of validity. That such a CFT should exist follows from the AdS/CFT correspondence.

In [11], using the same adiabatic type arguments employed to derive (4,4) CFT for freely acting Z_2 orbifold, we had obtained certain (4,0) symmetric product CFTs for freely acting orientifold models. These (4,0) CFTs gave the correct ground state multiplicities, however they were not in agreement with U-duality for excited states, if one assumes that the symmetric product CFT is defined for vanishing RR fields. However, we argued there that the symmetric product CFT is sitting at $\chi = 1/2$ component of the moduli space, where χ is the RR 0-form. In model II or in IIB on $K3$ (i.e. (4,4) models), χ is a modulus and therefore one can go continuously from $\chi = 1/2$ to $\chi = 0$ point. This is not the case for the (4,0) models appearing in the model III and type I'. In these cases χ is projected out and, as a result, $\chi = 0$ and $\chi = 1/2$ define two disconnected components of the moduli space. In [11], we presented an example to show that the physics of these two different components, such as the number of BPS states, can in general be very different. Now U-duality maps the $\chi = 1/2$ point of the (4,4) theory to the $\chi = 0$ component of the (4,0) theory, and not to the $\chi = 1/2$ component, where the symmetric product CFT exists. This could possibly explain the fact that the two symmetric product CFT s for (4,4) and (4,0) models did not match under U-duality map. The supergravity analysis, we have presented here, is presumably valid at $\chi = 0$. For the (4,4) case, as pointed out above, the physics at $\chi = 0$ should be the same as at $\chi = 1/2$, and therefore, the supergravity analysis agrees with the (4,4) symmetric product CFT. On the other hand, for the (4,0) case, there is no reason to expect that the elliptic genus for these two different components should be the same. In fact, from what has been said above, it is clear that the (4,0) supergravity elliptic genus at $\chi = 0$ should be related by U-duality to that of the (4,4) symmetric product CFT, as we indeed found in this paper. If this is the explanation for the apparent discrepancies, we must still look for the supergravity duals of the (4,0) symmetric product CFT s of [11], whose derivation was at the same level of rigour as the derivation of (4,4) models. This would, in particular, involve an understanding of how $\chi = 1/2$ background could modify the supergravity (or more precisely superstring on AdS_3) analysis.

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References

[1] J. Maldacena, *The large-N limit of superconformal field theories and supergravity*, *Adv. Theor. Math. Phys.* **2** (1998) 231 [hep-th/9711200]; *The large-N limit of su-*

perconformal field theories and supergravity, *Adv. Theor. Math. Phys.* **2** (1998) 231 [hep-th/9711200].

[2] C. Vafa, *Instantons on D-branes*, *Nucl. Phys.* **B 463** (1996) 435 [hep-th/9512078].

[3] A. Strominger and C. Vafa, *Microscopic origin of the Bekenstein-Hawking entropy*, *Phys. Lett.* **B 379** (1996) 99 [hep-th/9601029].

[4] E. Witten, *On the conformal field theory of the Higgs branch*, *J. High Energy Phys.* **07** (1997) 003 [hep-th/9707093].

[5] A. Giveon, D. Kutasov and N. Seiberg, *Comments on string theory on AdS_3*, *Adv. Theor. Math. Phys.* **2** (1998) 733 [hep-th/9806194].

[6] R. Dijkgraaf, *Instanton strings and hyperkaehler geometry*, *Nucl. Phys.* **B 543** (1999) 545 [hep-th/9810210].

[7] J. Maldacena and A. Strominger, *AdS_3 black holes and a stringy exclusion principle*, *J. High Energy Phys.* **12** (1998) 005 [hep-th/9804085].

[8] J. de Boer, *Six-dimensional supergravity on $S^3 \times AdS_3$ and 2D conformal field theory*, *Nucl. Phys.* **B 548** (1999) 139 [hep-th/9806104].

[9] J. de Boer, *Large-N elliptic genus and AdS/CFT correspondence*, *J. High Energy Phys.* **05** (1999) 017 [hep-th/9812240].

[10] J. Maldacena, G. Moore and A. Strominger, *Counting BPS black holes in toroidal type-II string theory*, hep-th/9903163.

[11] E. Gava, A.B. Hammou, J.F. Morales and K.S. Narain, *D1/D5 systems in $N = 4$ string theories*, hep-th/0012118.

[12] R. Dijkgraaf, G. Moore, E. Verlinde and H. Verlinde, *Elliptic genera of symmetric products and second quantized strings*, *Comm. Math. Phys.* **185** (1997) 197 [hep-th/9608096].

[13] A. Salam and J. Strathdee, *On Kaluza-Klein theory*, *Ann. Phys. (NY)* **141** (1982) 316

[14] S. Deger, A. Kaya, E. Sezgin and P. Sundell, *Spectrum of $D = 6$, $N = 4b$ supergravity on $AdS_3 \times S^3$*, *Nucl. Phys.* **B 536** (1998) 110 [hep-th/9804166].

[15] D. Kutasov and N. Seiberg, *More comments on string theory on AdS_3*, *J. High Energy Phys.* **04** (1999) 008 [hep-th/9903219].